Minimal configurations and sandpile measures

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Abstract

We give a new simple construction of the sandpile measure on an infinite graph G, under the sole assumption that each tree in the Wired Uniform Spanning Forest on G has one end almost surely. For, the so called, generalized minimal configurations the limiting probability on G exists even without this assumption. We also give determinantal formulas for minimal configurations on general graphs in terms of the transfer current matrix.

Key words: Abelian sandpile, sandpile measure, minimal configuration, uniform spanning tree, determinantal process

1 Introduction

In this paper we study minimal configurations and associated determinantal formulas in Abelian sandpiles. This will lead to a new simple construction of sandpile measures. Let us start by defining the Abelian sandpile model, deferring more detailed background to Section 2.

Let $G = (V \cup \{s\}, E)$ be a finite connected multigraph, with a distinguished vertex s, called the "sink". A *sandpile* on G is a configuration of particles on V, specified by a map $\eta : V \to \{0, 1, 2, \dots\}$, where $\eta(x)$ is the number of particles at x. If $\eta(x) \geq \deg_G(x)$, the vertex x can *topple* and send one particle along each edge incident with x. Particles that reach the sink are lost. A sandpile is *stable*, if no vertex can topple, that is, if $\eta(x) < \deg_G(x)$ for all $x \in V$.

We define a Markov chain on the set of stable sandpiles as follows. At each step, we add a particle at a uniformly random vertex of V, and carry out any possible topplings until a stable sandpile is reached. It was shown by Dhar [5] that

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the resulting stable sandpile does not depend on the order of topplings (Abelian property), and the stationary distribution is unique and uniformly distributed on the set of recurrent states. We denote the stationary distribution by ν_G or ν_V. Bak, Tang and Wiesenfeld [2] introduced the above model in a less general setting, prior to the work of Dhar, and hence the model is also known as the BTW sandpile. See the surveys [24] and [9] for background.

When $G = (V, E)$ is an infinite, locally finite, connected graph, and $V_1 \subset V$ is a finite set, we form the graph $G_{V_1} = (V_1 \cup \{s\}, E_{V_1})$, by identifying all vertices in $V \setminus V_1$ to a single vertex, that becomes the sink s of G_{V_1}, and removing loop-edges at s. It was shown by Athreya and Járai [1] that when $G = \mathbb{Z}^d$, $d \geq 2$, and $V_1 \subset V_2 \subset \cdots \subset \mathbb{Z}^d$ is a sequence of cubes with union $\mathbb{Z}^d$, then the stationary measures ν_{V_n} converge weakly to a limit ν, called the *sandpile measure* of $\mathbb{Z}^d$. This was generalized to certain other infinite graphs in [10]. Our main result in this paper will be a new simple construction of the sandpile measure ν on general graphs G, under the sole assumption:

$$\begin{array}{l}\text{each tree in the Wired Uniform Spanning} \\ \text{Forest on } G \text{ has one end almost surely.}\end{array} \tag{1}$$

The Wired Uniform Spanning Forest is a random spanning subgraph of G that is obtained, through a limiting process, from uniformly random spanning trees of finite graphs; see Section 2, where we also define the notion of "end". For certain special cylinder events, called generalized minimal subconfigurations, the limiting probability on G will be shown to exist even without assumption (1).

A result of Majumdar and Dhar [20] plays an important role in our proofs. These authors used the burning algorithm of Dhar [5] to construct a bijection between recurrent sandpiles on G and spanning trees of G. Since the stationary measure ν_G is uniform on the set of recurrent sandpiles, the burning bijection maps it to the uniform measure on spanning trees of G. This is known as the Uniform Spanning Tree measure; see e.g. [17] for background.

In many ways, the Uniform Spanning Tree is an easier object than recurrent sandpiles. One of its features that we use in this paper is that its marginal on a fixed set of edges is given by a simple determinantal formula. Let T_G denote a random spanning tree of G, chosen according to the uniform distribution. The Transfer Current Theorem of Burton and Pemantle [4] implies that the edges of T_G form a *determinantal process*. That is, there exists a matrix $Y_G(e, f)$, $e, f \in E$, the *transfer current matrix*, such that for any $k \geq 1$ and distinct edges $e_1, \dots, e_k \in E$ we have

$$\mathbf{P}[e_1, \dots, e_k \in T_G] = \det(Y_G(e_i, e_j))_{i,j=1}^k. \tag{2}$$

The matrix Y_G arises from a connection between spanning trees, electrical networks and random walk; see for example [3] for an exposition of these connections. For the sake of this introduction, we state the interpretation of Y_G in terms of random walk. Orient each edge in E arbitrarily. Given oriented edges e and f, consider simple random walk on G started at the tail of e and stopped at the first time it reaches the head of e. Let $J^e(f)$ denote the expected net usage of f, that is, the expected number of times the walk uses f, minus the expected number of times it uses the reversal of f. Then $Y_G(e, f) = J^e(f)$ can be taken as the definition of Y_G; see [7] or [3, Theorem 4.1]. Note that it is not difficult to see from this definition, using reversibility of the random walk, that the determinant on the right hand side of (2) does not depend on the chosen orientation of the edges. As pointed out in [4], the transfer current matrix can be expressed in terms of the Green function of simple random walk on G. In order to state this, first note that $J^e(f)$ is not affected if we replace the discrete time simple random walk by the continuous time simple random walk $\{S(t)\}_{t\geq 0}$ that crosses each edge at rate 1. (The generator of S is the negative of the graph Laplacian.) For vertices $x, y \in V \cup \{s\}$, let

$$H(x, y) := \lim_{t\to\infty} \mathbf{E}\Big[\int_0^t (I[S(t) = y] - I[S(t) = x])\, dt \Big| S(0) = x\Big].$$

The limit exists due to exponentially fast convergence to the uniform stationary distribution. If the tail and head of e are $x = \underline{e}$ and $y = \overline{e}$, and the tail and head of f are $u = \underline{f}$, $w = \overline{f}$, then it is not difficult to see that $J^e(f) = H(x, u) - H(y, u) - H(x, w) + H(y, w)$.

No simple expression similar to (2) is known for the marginal of ν_G on a fixed set of vertices. On the other hand, some determinantal formulas exist for special subconfigurations. Majumdar and Dhar [19] showed that on $\mathbb{Z}^d$, $d \geq 2$, the probability $p_0(d) = \nu[\eta : \eta(0) = 0]$ can be written as a determinant involving the simple random walk potential kernel $a(x) = H(0, x)$ ($d = 2$) or the Green function $G(x)$ ($d \geq 3$); see e.g. [14, Chapter 4] for the definitions of $a(x)$ and $G(x)$. In $d = 2$ the result is the explicit value $p_0(2) = \frac{2}{\pi^2} - \frac{4}{\pi^3}$. Majumdar and Dhar also showed that in dimensions $d \geq 2$, the correlation between the events of seeing no particle at x and y, respectively, decays as

$$\nu[\eta(x) = 0, \eta(y) = 0] - p_0(d)^2 \sim c|x - y|^{-2d}, \quad \text{as } |x - y| \to \infty.$$

More generally, the probability of the event that none of the vertices $x_1, \dots, x_k$ has a particle, is given by a "block-determinantal" formula [19]. This can be written as

$$\nu[\eta(x_1) = 0, \dots, \eta(x_k) = 0] = \det(M(i, j))_{i,j=1}^k.$$

where, in its most reduced form, each $M(i, j)$ is a $(2d - 1) \times (2d - 1)$ matrix block. The above explicit form was exploited by Dürre [8], who gave rigorous scaling limit results in 2D for the random field of vertices having no particles.

In its most general form, the method of Majumdar and Dhar applies to *minimal subconfigurations.* We say that a stable configuration ξ of particles on a subset $W \subset V$ is *minimal*, if it has an extension to a recurrent sandpile on V, but removing a particle from any of the vertices in W would render such an extension impossible. (In fact, for technical reasons, we are going to distinguish between *minimal* and *generalized minimal* subconfigurations, but this distinction can be ignored for now.)

The computations quoted above are all examples of the form $W = \{x_1, \dots, x_k\}$, $\xi(x_i) = 0$, $i = 1, \dots, k$. In Theorems 1 and 2 below, we formulate the general statement that the probability of any minimal subconfiguration can be written as a determinant involving the transfer current matrix. This type of result appears to be taken for granted in the physics literature, see e.g. [18]. Yet, we have not found it clearly stated anywhere in a general form, and it seems worthwhile to record it here. Our formulation in terms of the transfer current matrix is slightly different from what is usually used in the physics community. We believe that our formulation highlights what makes the theorem work: namely that minimal events can be expressed, via the burning bijection, as the absence of a fixed set of edges from the Uniform Spanning Tree.

Consider now the probabilities of the non-minimal events:

$$p_i(d) := \nu[\eta : \eta(0) = i], \quad i = 1, \dots, 2d-1, \quad d \geq 2,$$

where ν is the sandpile measure on $\mathbb{Z}^d$. Majumdar and Dhar [19, Eqn. (14)] gave an infinite series for the value of $p_1(2)$ where each term in the series can be written as a determinant. A rigorous proof that the series indeed converges to $p_1(2)$ can be given based on the fact that assumption (1) is satisfied for $\mathbb{Z}^d$, $d \geq 2$. A similar series can be given for $p_i(d)$ in general. It was by considering extensions of the series of Majumdar and Dhar that we arrived at our main result, Theorem 3, saying that under assumption (1), a unique sandpile measure ν exists. Some of the arguments of Levine and Peres [15] were also inspiring, who prove fascinating connections between the average number of particles $\sum_{i=0}^{d-1} i p_i(d)$, and seemingly unrelated constants in other models.

Construction of the sandpile measure ν is the first step in understanding the asymptotic behaviour of the model on a growing sequence of subgraphs of an infinite graph G. Hitherto ν has only been shown to exist under more restrictive assumptions on G, such as connectedness of the Wired Uniform Spanning Forest, or for certain transitive graphs; see [1, 10]. We believe that the greater generality of our theorem will be useful in studying Abelian sandpiles on some irregular graphs, for example, graphs obtained as the result of a random process. We know that assumption (1) cannot be omitted: Járai and Lyons [11] show that on graphs of the form $G = \mathbb{Z} \times G_0$, where G_0 is any connected finite graph with at least two vertices, there are two distinct ergodic weak limit points of the family $\{\nu_U : U \subset V(G),\, U \text{ finite}\}$. The following question, that is a strengthened form of a question of [11], remains open, even in the case of $\mathbb{Z}^2$.

Open Question 1. Assume that G is recurrent, satisfies assumption (1), and o is a fixed vertex of G. Draw a configuration from the measure ν, add a particle at o, and carry out all possible topplings. Is it true that there are finitely many topplings ν-a.s.?

Note that the statement holds when G is transient and satisfies assumption (1); this follows by the argument of [12, Theorem 3.11].

In Section 2 we give further definitions and state our results. In Section 3 we collect preliminary results. In Section 4 we give the general construction of sandpile measures.

A brief announcement of our results appeared in the proceedings [13].

2 Definitions and main results

Let $G = (V \cup \{s\}, E)$ be a finite connected multigraph. We write b_{uv} for the number of edges between vertices u and v, and write $u \sim v$ if $b_{uv} \geq 1$. We write $\mathcal{R}_G$ for the set of recurrent sandpiles on G. Let $F \subset V$. We say that the stable sandpile η on G is *ample* for $F \subset V$, if there exists $x \in F$ such that $\eta(x) \geq \deg_F(x)$, where $\deg_F(x) = \#\{y \in F : x \sim y\}$. (Here $\#A$ denotes the number of elements of A.) By the well-known burning test of Dhar [5] (see also [9, Lemma 4.2]), we have

$$\eta \in \mathcal{R}_G \qquad \text{if and only if} \qquad \eta \text{ is ample for every nonempty } F \subset V. \tag{3}$$

Let $W \subset V$. We define the graph G_W by "wiring the complement of W", i.e. identifying all vertices in $V \setminus W$ with s, and removing loop-edges. Due to the criterion (3), the restriction of any $\eta \in \mathcal{R}_G$ to W, denoted η_W, is in $\mathcal{R}_{G_W}$. When the choice of G is clear from the context, we write $\mathcal{R}_W$ for $\mathcal{R}_{G_W}$.

The burning bijection [20] establishes a one-to-one mapping between recurrent sandpiles and spanning trees of G. We will need a particular version of this bijection that is introduced in Section 3.1.

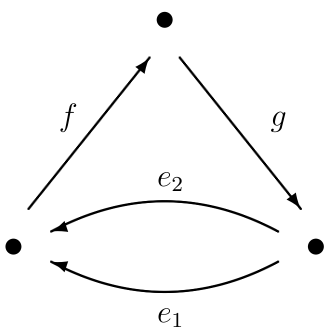
There is a relationship between spanning trees and electrical networks, for which we refer the reader [17] or [3, Section 4]. The *transfer current matrix* Y_G is defined by regarding G as an electrical network, as follows. Choose an orientation for each edge of G. Replace each edge of G by a wire of unit resistance, and hook up a battery between the two endpoints of e. Suppose that the voltage of the battery is such that in the network as a whole, unit current flows from the tail of e to the head of e. Let $I^e(f)$ be the amount of current flowing through edge f consistent with its orientation. (Hence $I^e(f)$ can be positive or negative depending on whether the current flows in the same direction or not as the orientation of f). The matrix $Y_G(e, f) := I^e(f)$, indexed by the edges of G, is the *transfer current matrix.* (See [7] for a proof that $I^e(f) = J^e(f)$.) An example is given in Figure 1.

We define the matrix

$$K_G(e, f) := \delta_G(e, f) - Y_G(e, f),$$

where δ_G is the identity matrix. An extension of the Transfer Current Theorem [4, Corollary 4.4] implies that if $e_1, \dots, e_k$ are distinct edges and T_G is a uniformly random spanning tree of G, then

$$\mathbf{P}[e_1, \dots, e_k \notin T_G] = \det(K_G(e_i, e_j))_{i,j=1}^k. \tag{4}$$



$$Y_G = \begin{pmatrix} 2/5 & 2/5 & -1/5 & -1/5 \\ 2/5 & 2/5 & -1/5 & -1/5 \\ -1/5 & -1/5 & 3/5 & -2/5 \\ -1/5 & -1/5 & -2/5 & 3/5 \end{pmatrix}$$

Figure 1: Example of a transfer current matrix. The columns correspond to the edges in the order e_1, e_2, f, g. The entries can be computed by simple applications of the series-parallel laws. There are 5 spanning trees.

We always regard T_G as a rooted tree, with root at s. By analogy with family trees, we call a vertex x a *descendent* of the vertex y, if y lies on the unique path from x to s (here we allow $x = y$).

The above can be extended to a locally finite, connected, infinite graph $G = (V, E)$ as follows. By an exhaustion of G we mean a sequence of finite subsets $V_1 \subset V_2 \subset \cdots \subset V$ such that $\cup_{n \geq 1} V_n = V$. Form the finite graphs $G_n := G_{V_n} = (V_n \cup \{s\}, E_n)$ by identifying $V \setminus V_n$ to a single vertex s, and removing loops at s. Note that each edge in E_n can be regarded, in a natural way, as an element of E. Let μ_n denote the probability measure on $\{0,1\}^{E_n}$ that is supported on spanning trees of G_n and gives equal weight to each spanning tree. Here the value 1 corresponds to an edge being present, and we identify any spanning subgraph of G_n with the set of edges it contains. We write T_{G_n} for the random variable on $\{0,1\}^{E_n}$ that is the collection of edges contained in the corresponding spanning subgraph. We similarly define the random variable T_G on the space $\{0,1\}^E$. By a result of Pemantle [22] (see also [3]), the weak limit $\mu = \lim_{n\to\infty} \mu_n$ exists, as a measure on $\{0,1\}^E$, and is independent of the exhaustion. That is, for any finite sets $B, K \subset E$ we have

$$\mu[T_G \cap K = B] = \lim_{n\to\infty} \mu_n[T_{G_n} \cap K = B].$$

The measure μ is called the *Wired Uniform Spanning Forest measure* of G (we henceforth abreviate this to WSF). The term "wired" refers to the particular choice of boundary condition, that is, the identification of vertices in $V \setminus V_n$. The measure μ is supported on spanning subgraphs of G, each of whose components is an infinite tree.

A *ray* in an infinite tree is an infinite self-avoiding path. An *end* of an infinite tree is defined as an equivalence class of rays, where two rays are equivalent, if they have a finite symmetric difference. Hence an infinite tree has one end if and only if it contains no two disjoint infinite self-avoiding paths.

The *wired current* in G is the pointwise limit $I^e = \lim_{n\to\infty} I^e_{V_n}$ that exists by monotonicity; see e.g. [3]. Both (2) and (4) have limiting versions on G [4, Theorem 4.2], involving $Y_G(e,f) = I^e(f) = \lim_{n\to\infty} Y_{G_n}(e,f)$ and $K_G(e,f) = \lim_{n\to\infty} K_{G_n}(e,f)$.

Properties of the WSF have been studied extensively. Pemantle [22] proved that on $\mathbb{Z}^d$, $2 \le d \le 4$, T_G is a single tree and has one end μ-a.s. He also proved that when $d \ge 5$, T_G consists of infinitely many trees, and each has one or two ends μ-a.s. This was completed and extended by Benjamini, Lyons, Peres and Schramm [3], who showed in particular that in the case $d \ge 5$ each tree has one end μ-a.s. In fact, their general result (Theorem 10.1 in the reference above) implies that on any Cayley graph that is not a finite extension of $\mathbb{Z}$, each tree in T_G has one end μ-a.s. Examples of graphs where T_G is connected and has two ends μ-a.s. are provided by graphs of the form $G = \mathbb{Z} \times G_0$, where G_0 is a finite, connected graph; see [3, Proposition 10.10] for a more general result. Lyons, Morris and Schramm [16] gave a very general condition on the isoperimetric profile of a graph that ensures that each tree in T_G has one end μ-a.s. From the above it is clear that assumption (1) is known to hold on many graphs.

We regard any one-ended infinite tree as being "rooted at infinity", and we call vertex x a *descendent* of vertex y, if y lies on the (necessarily unique) infinite self-avoiding path in the tree that starts at x. (Here we allow $x = y$.)

The starting point for this paper is the notion of a minimal configuration. It is well-known, and easy to see using (3), that if η is a recurrent sandpile on G, then adding more particles to η also results in a recurrent sandpile, as long as it remains stable. Let δ_u denote the sandpile with a single particle at u and no other particles.

Definition 1. Let $G = (V \cup \{s\}, E)$ be a finite, connected multigraph and let $\emptyset \ne W \subset V$ be such that $G \setminus W$ (the graph obtained from G by removing the vertices in W) is connected. Let ξ be a stable configuration on W. We say that ξ is *minimal*, if there exists a recurrent sandpile $\eta \in \mathcal{R}_G$ such that $\eta_W = \xi$, and for any such η and any $w \in W$, we have $\eta - \delta_w \notin \mathcal{R}_G$.

Remark 1. Equivalently, it is enough to require that the sandpile

$$\eta^*(x) = \begin{cases} \xi(x) & \text{if } x \in W; \\ \deg_G(x) - 1 & \text{if } x \in V \setminus W; \end{cases} \tag{5}$$

is in $\mathcal{R}_G$ and for all $w \in W$ we have $\eta^* - \delta_w \notin \mathcal{R}_G$. This implies that ξ is minimal if and only if $\xi \in \mathcal{R}_{G_W}$, but for any $w \in W$ we have $\xi - \delta_w \notin \mathcal{R}_{G_W}$.

Definition 2. When the restriction that $G \setminus W$ be connected is dropped, and ξ satisfies the requirements of Definition 1, we say that ξ is *generalized minimal*. (That is, in this case we allow W to have "holes").

We extend Definition 1 to infinite graphs as follows.

Definition 3. Let $G = (V, E)$ be a locally finite, connected, infinite graph, and let $W \subset V$ be a finite set such that all connected components of $G \setminus W$ are infinite. Let ξ be a stable configuration on W. We say that ξ is *minimal*, if for some (and then for any) finite $V_1 \supset W$ the configuration ξ is minimal in the graph G_{V_1}. When $G \setminus W$ is allowed to have finite components, we call ξ *generalized minimal*, if it is generalized minimal with respect to some (and then for any) finite $V_1 \supset W$ for which $G \setminus V_1$ has only infinite components.

Theorems 1 and 2 below summarizes what can be proved using the method of Majumdar and Dhar [19]. The extension to generalized minimal configurations appear to be new. Let Δ_G denote the graph Laplacian on G, that is:

$$\Delta_G(x, y) = \begin{cases} \deg_G(x) & \text{if } x = y; \\ -b_{xy} & \text{if } x \sim y; \\ 0 & \text{otherwise;} \end{cases} \qquad x, y \in V;$$

where $x \sim y$ denotes that x and y are adjacent.

Theorem 1. *Let $G = (V \cup \{s\}, E)$ be a finite, connected multigraph, and let ξ be minimal on $W \subset V$. There exists a subset $\mathcal{E}$ of the set of edges touching W such that*

$$\nu_G[\eta : \eta_W = \xi] = \det(K_G(e, f))_{e,f \in \mathcal{E}}. \tag{6}$$

The statement remains true for generalized minimal configurations.

Remark 2. Alternatively, following the arguments of [19], the matrix can be replaced by some $R_{G,W}$ whose entries can be expressed in terms of $\Delta_G^{-1}(x, y)$, $x, y \in W \cup \partial_{\text{ext}} W$, with $\partial_{\text{ext}} W = \{y \in V \setminus W : \exists\, x \in W,\, x \sim y\}$.

Theorem 2. *Let $G = (V, E)$ be a locally finite, connected, infinite graph, and $V_1 \subset V_2 \subset \cdots \subset V$ any exhaustion. Let $W \subset V$ be finite, and let ξ be minimal on W. There exists a subset $\mathcal{E}$ of the set of edges touching W such that*

$$\lim_{n \to \infty} \nu_{V_n}[\eta : \eta_W = \xi] = \det(K_G(e, f))_{e,f \in \mathcal{E}}. \tag{7}$$

The statement remains true for generalized minimal configurations.

Remark 3. When $G = \mathbb{Z}^d$, $d \geq 2$, the matrix can be replaced by some R_W with entries expressed in terms of $a(x)$ or $G(x)$.

We now state our general construction of sandpile measures.

Theorem 3. *Let $G = (V, E)$ be a locally finite, connected, infinite graph. Suppose that G satisfies the one-end property (1). There exists a unique measure ν on the space $\prod_{x \in V} \{0, \ldots, \deg_G(x) - 1\}$ such that along any exhaustion $V_1 \subset V_2 \subset \cdots \subset V$ the measures ν_{V_n} converge weakly to ν.*

Our proof of Theorem 3 is much simpler than earlier proofs in [1, 10] in more restrictive settings. We note however, that unlike the proof of Theorem 3, the earlier proofs do give more than weak convergence, as they construct the sandpile measure as the image, under a measurable map, of the WSF with extra randomness.

The proof of Theorem 3 exhibits the limiting probability of a cylinder event as an infinite series. In the case of the event $\{\eta : \eta(o) = k\}$ with some $k = 0, \dots, \deg_G(o) - 1$, the series is a generalized version of [19, Eqn. (14)]. The decomposition of the event into this series is also implicit in [23, Section 3].

3 Preliminaries

3.1 The burning bijection

In this section $G = (V \cup \{s\}, E)$ is a finite, connected multigraph with sink s. Fix $Q \subset V$. We consider a particular version of the burning bijection [20] depending on Q, for sandpiles on V. For the reader familiar with the usual construction, we note that the idea is to burn in two phases: first we burn all vertices we can without touching the set Q, then the remaining vertices.

Fix a stable sandpile η on V, and let

$$B^{(1)}_{Q,0} = \{s\} \qquad \text{and} \qquad U^{(1)}_{Q,0} = V,$$

and for $i \geq 1$ define inductively

$$\begin{aligned} B^{(1)}_{Q,i} &= \left\{ x \in U^{(1)}_{Q,i-1} \setminus Q : \eta(x) \geq \deg_{U^{(1)}_{Q,i-1}}(x) \right\} \\ U^{(1)}_{Q,i} &= U^{(1)}_{Q,i-1} \setminus B^{(1)}_{Q,i}. \end{aligned}$$

We call $B^{(1)}_{Q,i}$ the vertices that burn at time i in the first phase, and $U^{(1)}_{Q,i}$ the vertices that remained unburnt at time i. There exists a smallest index I such that for $i \geq I$ we have $U^{(1)}_{Q,i+1} = U^{(1)}_{Q,i}$. To define the second phase, set

$$B^{(2)}_{Q,0} = \bigcup_{0 \leq i \leq I} B^{(1)}_{Q,i} = V \setminus U^{(1)}_{Q,I} \qquad \text{and} \qquad U^{(2)}_{Q,0} = U^{(1)}_{Q,I}.$$

Then for $i \geq 1$ we set

$$\begin{aligned} B^{(2)}_{Q,i} &= \left\{ x \in U^{(2)}_{Q,i-1} : \eta(x) \geq \deg_{U^{(2)}_{Q,i-1}}(x) \right\} \\ U^{(2)}_{Q,i} &= U^{(2)}_{Q,i-1} \setminus B^{(2)}_{Q,i}. \end{aligned}$$

It is not difficult to show using (3) that $U^{(2)}_{Q,i} = \emptyset$ eventually if and only if $\eta \in \mathcal{R}_V$.

We now define the burning bijection corresponding to the above burning rule. Fix for each $x \in V$ an ordering $<_x$ of the edges incident with x in the graph G_V.

We assign to $\eta \in \mathcal{R}_V$ a spanning tree t of G_V, by specifying for each $x \in V$ an oriented edge e_x with $\underline{e_x} = x$ to be an edge of t (here we write $\underline{e}$ for the tail of e and $\overline{e}$ for the head of e). If $x \in B^{(1)}_{Q,i}$ for $i \geq 1$, then let

$$n_x = \sum_{y \in \cup_{0 \leq j < i} B^{(1)}_{Q,j}} b_{xy}$$

$$P_x = \left\{ e : \underline{e} = x,\, \overline{e} \in B^{(1)}_{Q,i-1} \right\} =: \{e_0 <_x \cdots <_x e_{|P_x|-1}\}.$$

It follows from the burning rules that $\eta(x) = 2d - n_x + i$ for some $0 \leq i < |P_x|$, and hence we can define e_x to be the i-th element of P_x in the order $<_x$. If $x \in B^{(2)}_{Q,i}$ for some $i \geq 1$, we make exactly the same definitions replacing the superscript (1) with (2). It follows easily that t is a spanning tree of G. Note that we defined t using directed edges, and this way each edge of t is directed towards s. It is somewhat tedious, but fairly straightforward to check that the map is a bijection. We refer to this as the *"bijection based on Q"*. When in the above $Q = \emptyset$, we only have phase 1, and we refer to this as the *usual bijection*. Observe that there is in fact much more flexibility in choosing the ordering $<_x$ than we stated above. For example, we can allow the choice of $<_x$ to depend on the set of vertices burnt up to the time when x is burnt. We will freely make use of this flexibility in the sequel.

A special role will be played by the event that all vertices in $V \setminus Q$ can be burnt in the first phase, that is the event

$$E_{V,Q} = \{\eta \in \mathcal{R}_V : U^{(1)}_{Q,I}(\eta) = Q\}. \tag{8}$$

Under the bijection based on Q, this corresponds to the event that in t there is no directed edge pointing from $V \setminus Q$ to Q.

3.2 Minimal subconfigurations

Lemma 4. *Let $G = (V \cup \{s\}, E)$ be a finite, connected multigraph, $W \subset V$ and let ξ be minimal on W. For any $\eta \in \mathcal{R}_G$ such that $\eta_W = \xi$, we have $\eta \in E_{V,W}$, i.e. there is a burning sequence that burns all of $V \setminus W$ before burning any vertex in W.*

Proof. We argue by contradiction. Suppose that $\eta \in \mathcal{R}_G$, $W \subsetneq U \subset V$, all of $V \setminus U$ can be burnt before burning any vertex of U, but no further vertex of $U \setminus W$ can be burnt without touching W. Let $v_1, v_2, \dots$ be a possible continuation of the burning of η in U. In particular, $v_1 \in W$. Let $i \geq 1$ be the smallest index such that $v_1, v_2, \dots, v_i \in W$ and v_i neighbours a vertex in $U \setminus W$. Such an index has to exist, since there will be a first time when a vertex of $U \setminus W$ becomes burnable. Consider now $\xi' = \xi - \delta_{v_i}$. The sequence $v_1, v_2, \dots, v_i$ is a burning sequence for ξ' that removes the vertex v_i, and it follows that $\xi' \in \mathcal{R}_{G_W}$. Recall that $G \setminus W$ is connected, so it follows that with η^* defined as in (5), we have $\eta^* - \delta_{v_i} \in \mathcal{R}_G$.

This contradicts the assumption that ξ is minimal, and hence the statement of the Lemma follows. $\square$

The next lemma gives a recursive characterization of minimal sandpiles.

Definition 4. Let $W \subset V$ such that $G \setminus W$ is connected. The *entry points* of $\xi \in \mathcal{R}_{G_W}$ are the vertices $E(\xi, W) = \{w_1, \dots, w_k\} \subset W$ that are burnable for ξ in W at the first step of the burning algorithm.

Lemma 5. *Let $W \subset V$ such that $G \setminus W$ is connected. Suppose that $\xi \in \mathcal{R}_{G_W}$ is minimal, with entry points $E(\xi, W) = \{w_1, \dots, w_k\}$.*
(i) If $1 \le i < j \le k$, then $w_i \not\sim w_j$.
(ii) For each $i = 1, \dots, k$ we have $\xi(w_i) = \sum_{v \in W : v \sim w_i} b_{vw_i}$.
(iii) The subconfiguration $\xi_{W \setminus E(\xi, W)}$ is minimal.
Conversely, if $\xi \in \mathcal{R}_{G_W}$ satisfies (i)–(iii), then it is minimal.

Proof. (i) Suppose we had $w_i \sim w_j$. Then there is a burning sequence for ξ starting with w_1, w_2, and hence $\xi(w_2) \ge \sum_{v \in W : v \sim w_2,\, v \ne w_1} b_{vw_2}$. Due to minimality, we must have $\xi(w_2) = \sum_{v \in W : v \sim w_2,\, v \ne w_1} b_{vw_2}$. But this contradicts the assumption that $w_2 \in E(\xi, W)$.

(ii) Since w_i is burnable in ξ, we must have $\xi(w_i) \ge \sum_{v \in W : v \sim w_i} b_{vw_i}$. Again, due to minimality, we must have equality here.

(iii) Burning all vertices in $E(\xi, W)$ gives a configuration in $\mathcal{R}_{W \setminus E(\xi, W)}$. This configuration also has to be minimal, as otherwise ξ would not be minimal.

Suppose now that $\xi \in \mathcal{R}_W$ satisfies (i)–(iii). Due to (iii), for any $w \in W \setminus E(\xi, W)$ the configuration $\xi - \delta_w$ is not ample for some subset of $W \setminus E(\xi, W)$. Let now $w \in E(\xi, W)$ and consider $\xi' = \xi - \delta_w$. Write $E(\xi, W) = \{w_1, \dots, w_k\}$, and assume the indexing is such that $w = w_k$. In order to arrive at a contradiction assume that $\xi' \in \mathcal{R}_W$. Note that $w = w_k$ is not burnable in ξ' at the first step of the burning algorithm, while the vertices $w_1, \dots, w_{k-1}$ are all still burnable in ξ' at the first step. Hence there exists a burning sequence for ξ' of the form

$$v_1 = w_1, \dots, v_{k-1} = w_{k-1}, v_k, \dots, v_l = w_k, \dots, v_K$$

where $W \setminus E(\xi, W) = \{v_k, \dots, v_K\} \setminus \{v_l\}$. Let $i \ge 1$ be the first index such that $v_i \sim w_k$. Note that necessarily $i < l$, and due to (i), $i \ge k$. Since at the time of burning of v_i the vertex w_k has not been burnt yet, the sequence

$$v_k, \dots, v_i, \dots, v_{l-1}, v_{l+1}, \dots, v_K$$

is a burning sequence for $\xi_{W \setminus E(\xi, W)} - \delta_{v_i}$. This contradicts assumption (iii), and hence the proof is complete. $\square$

Remark 4. As a corollary, we obtain by induction the well known fact that all minimal configurations on W have the same total number of particles and this equals the number of edges of G_W minus the degree of s in G_W; see [21, Theorem 3.5] and [6].

We next describe a burning procedure we can apply to generalized minimal configurations. Let $W \subset V$, suppose that ξ is generalized minimal on W, and $\eta \in \mathcal{R}_G$ such that $\eta_W = \xi$. Let $V_1, \dots, V_K$ be the connected components of $G \setminus W$ not containing s. Let $W' := W \cup V_1 \cup \dots \cup V_K$. The burning of η is defined in several phases.

Phase 1. By the same argument as Lemma 4 we get that $E_{V,W'}$ occurs, so in Phase 1 we burn all of $V \setminus W'$.

Phase 2. Burn vertices in W as in usual burning, until the first time that a vertex neighbouring some V_j becomes burnable. Let $y_1^{(j)}, \dots, y_{r_j}^{(j)}$ be the vertices in W neighbouring V_j that became burnable at this stage.

Lemma 6. *If $r_j \geq 1$, then after burning $y_1^{(j)}$, all vertices in V_j can be burnt, without touching W.*

Lemma 7. *For each $1 \leq j \leq K$ we have $r_j = 0$ or 1.*

We prove these lemmas after we completed the definition of the burning process.

Phase 3. For each j such that $r_j = 1$, burn $y_1^{(j)}$ and then burn all of V_j, appealing to Lemmas 6 and 7. Without loss of generality we assume that the V_j's that were *not* burnt are $V_1, \dots, V_{K_1}$ for some $0 \leq K_1 < K$.

Following this we iterate Phases 2 and 3 for the remaining vertices.

We use the above process to define an auxiliary graph G_W^*. This is obtained from $G_{W'}$ by identifying all vertices in V_j with $y_1^{(j)}$, $1 \leq j \leq K$, and removing loop-edges. Then ξ viewed as a configuration on G_W^* is recurrent and minimal.

Let us now prove the two lemmas.

Proof of Lemma 6. This is similar to the proof of Lemma 4. Suppose there is a non-empty subset $U \subset V_j$ such that all of $V_j \setminus U$ can be burnt without touching W, but no further vertex of U can be burnt. Let $v_1, v_2, \dots$ be a continuation of the burning of $\eta_{W'}$. There is a first index i such that v_i neighbours a vertex in U. Then we can apply the same sequence to $\eta^* - \delta_{v_i}$, and see that this is in $\mathcal{R}_G$. Since $v_i \in W$, this contradicts that ξ was generalized minimal. $\square$

Proof of Lemma 7. This is similar to the proof of Lemma 5(i). Suppose we have $r_j \geq 2$ for some j. Burn $y_1^{(j)}$, and then all of V_j, using Lemma 6. Since $y_2^{(j)}$ neigbours V_j, this shows that we can decrease the number of particles at $y_2^{(j)}$, contradicting the minimality of ξ. $\square$

We now prove Theorem 1.

Proof of Theorem 1. By Lemma 4, the event $\eta_W = \xi$ implies the event $E_{V,W}$. Using the burning rule based on W, the conditional distribution of η_W given the event $E_{V,W}$ is uniform on $\mathcal{R}_W$. Hence

$$\nu_G[\eta : \eta_W = \xi] = \nu_G[E_{V,W}](\det(\Delta_{G_W}))^{-1}. \tag{9}$$

In order to write this in terms of the transfer current matrix, choose any spanning tree t_0 of G_W (e.g. the tree corresponding to ξ under the burning bijection in the graph G_W), and let

$$\mathcal{E} := \{\{x, y\} : x \in W,\ \{x, y\} \notin t_0\}.$$

Then letting t denote the tree corresponding to η under the burning bijection based on W in the graph G, the event $\{\mathcal{E} \cap t = \emptyset\}$ is equivalent to the event that $E_{V,W}$ occurs, and η_W is a fixed element of $\mathcal{R}_W$. Hence by (4), the right hand side of (9) equals

$$\det(K_G(e, f))_{e,f\in\mathcal{E}}.$$

Assume now that ξ is generalized minimal. Consider the auxiliary graph G^*_W constructed earlier. Let t_0 be the spanning tree assigned to ξ under the bijection in the graph G^*_W. Let $\mathcal{E}^*$ be the set of edges of G^*_W not present in t_0. To each edge of $\mathcal{E}^*$ corresponds an edge of G touching W, let us call these edges $\mathcal{E}$. Then the identification of the burning processes on G and G^*_W shows that $\{\eta_W = \xi\}$ is equivalent to $\{\mathcal{E} \cap T_G = \emptyset\}$, where T_G is the Uniform Spannign Tree on G. Hence the statement follows with the set $\mathcal{E}$. □

Proof of Theorem 2. This is immediate from the proof of Theorem 1, as the event $T_{G_n} \cap \mathcal{E} = \emptyset$ is a cylinder event, and hence its probability converges, as $n \to \infty$, to $\mu[T_G \cap \mathcal{E} = \emptyset]$. □

4 A general construction of sandpile measures

In this section we prove Theorem 3. If t is a spanning tree of G_{V_n}, and $x, y \in V_n$, we say that x is a *descendent* of y, if y lies on the unique directed path from x to s (allowing $x = y$ as well).

Proof of Theorem 3. We need to show that for any fixed finite set $Q \subset V$ and $\xi \in \mathcal{R}_Q$ the probabilities $\nu_{V_n}[\eta_Q = \xi]$ have a limit as $n \to \infty$. Without loss of generality assume that all components of $G \setminus Q$ are infinite. Let $\eta \in \mathcal{R}_{V_n}$ and let $W_0(\eta) \supset Q$ be the set of vertices that do not burn in the first phase, under the bijection based on Q.

Fix $Q \subset W \subset V_n$, and assume the event $W_0(\eta) = W$. We define an auxiliary graph $G^*_W = (W \cup \{s\}, E^*_W)$ as follows. All edges of G_{V_n} between vertices $x, y \in W$ are also present in E^*_W. For every edge $\{x, y\} \in E_{V_n}$ satisfying $x \in Q$, $y \in (V_n \cup \{s\}) \setminus W$ we place an edge between x and s in E^*_W. There are no other edges in E^*_W.

We claim that

$$W_0(\eta) = W,\ \eta_Q = \xi \qquad \text{if and only if} \qquad \eta \in E_{V_n,W},\ \eta_W \in \mathcal{R}_{G^*_W},\ \eta_Q = \xi. \quad (10)$$

To see this, assume first the left hand statement. It is clear that $W_0(\eta) = W$ implies $E_{V_n,W}$, hence we only need to prove that $\eta_W \in \mathcal{R}_{G^*_W}$. It is clear that

$$\eta(x) < \deg_G(x) = \deg_{G^*_W}(x) \quad \text{for } x \in Q.$$

Also, using that a vertex $x \in W \setminus Q$ does not burn in phase one, we have

$$\eta(x) < \deg_W(x) = \deg_{G^*_W}(x) \quad \text{for } x \in W \setminus Q.$$

Therefore, η_W is a stable sandpile on G^*_W. Now it follows easily from the fact that η_W burns in phase two that η_W also burns in the graph G^*_W and hence $\eta_W \in \mathcal{R}_{G^*_W}$.

Now assume the right hand statement in (10). Then we know that all vertices in $V_n \setminus W$ can be burnt without touching W, and in particular, without touching Q, so $W_0(\eta) \subset W$. However, $\eta_W \in \mathcal{R}_{G^*_W}$ implies that for $x \in W \setminus Q$ we have $\eta(x) < \deg_W(x)$, and hence no more vertices can be burnt in the first phase, implying that $W_0(\eta) = W$. This proves (10).

The equivalence (10) hence gives the following decomposition:

$$\begin{aligned}\nu_{V_n}[\eta_Q = \xi] &= \sum_{W: Q \subset W \subset V_n} \nu_{V_n}[W_0(\eta) = W,\, \eta_Q = \xi] \\ &= \sum_{W: Q \subset W \subset V_n} \nu_{V_n}[E_{V_n,W}]\, \nu_{G^*_W}[\eta_Q = \xi].\end{aligned}$$

It follows from the observation made after (8), that the event $E_{V_n,W}$ is spanning-tree-local, that is, it only depends on the status of the edges touching W. Hence for fixed W we have

$$\nu_{V_n}[E_{V_n,W}] \overset{n\to\infty}{\longrightarrow} \text{some limit } p_W.$$

Hence we get

$$\lim_{n\to\infty} \nu_{V_n}[\eta_Q = \xi] = \sum_{\substack{W: Q \subset W \subset V \\ W \text{ finite}}} p_W \nu_{G^*_W}[\eta_Q = \xi],$$

provided we can show that for any $\varepsilon > 0$ there exists a finite $V_0 \subset V$ such that

$$\sup_{n\geq 1} \nu_{V_n}[W_0 \not\subset V_0] < \varepsilon. \tag{11}$$

In order to show (11), we observe that under the bijection based on Q, for every $\eta \in \mathcal{R}_{V_n}$, $W_0(\eta)$ contains precisely those vertices that are descendents of some vertex of Q in $t = t(\eta)$. Recall that due to the assumed one end property of the WSF on G, the notion of "descendent" extends to the infinite case: x is a descendent of y if and only if y lies on the unique self-avoiding path from x to infinity. Let us write $\mathcal{D}(Q)$ for the set of descendents of Q. We have $\mu[|\mathcal{D}(Q)| < \infty] = 1$, hence there exists a finite $V'_0 \subset V$ such that $\mu[\mathcal{D}(Q) \not\subset V'_0] < \varepsilon$. Since μ_{V_n} converges weakly to μ and for fixed W the event $\mathcal{D}(Q) = W$ is a cylinder event, we have $\mu_{V_n}[\mathcal{D}(Q) \not\subset V'_0] < \varepsilon$ for all large enough n. Taking V_0 suitably larger than V'_0 we get $\mu_{V_n}[\mathcal{D}(Q) \not\subset V_0] < \varepsilon$ for all $n \geq 1$. This proves (11), and completes the proof of the theorem. $\square$

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